Polynomial Solution of $\mathcal{PT}$/Non-$\mathcal{PT}$-Symmetric and Non-Hermitian Generalized Woods-Saxon Potential via Nikiforov-Uvarov Method

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Abstract

Using the Nikiforov-Uvarov method, the bound state energy eigenvalues and eigenfunctions of the $\mathcal{PT}$-/non-$\mathcal{PT}$-symmetric and non-Hermitian generalized Woods-Saxon (WS) potential with the real and complex-valued energy levels are obtained in terms of the Jacobi polynomials. According to the $\mathcal{PT}$-symmetric quantum mechanics, we exactly solved the time-independent Schrödinger equation with same potential for the s-states and also for any l-state as well. It is shown that the results are in good agreement with the ones obtained before.

Keywords: Energy Eigenvalues and Eigenfunctions; Generalized Woods-Saxon Potential; $\mathcal{PT}$-symmetry, Non-Hermitian potential; Nikiforov-Uvarov Method.

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I. INTRODUCTION

A large variety of potentials with the real or complex form are encountered in various fields of physics. A consistent physical theory of quantum mechanics in terms of Hermitian Hamiltonians is built on a complex Hamiltonian that is not Hermitian, but the energy levels are real and positive as a consequence of $\mathcal{PT}-$symmetry. By definition, a $\mathcal{PT}-$symmetric Hamiltonian H satisfies $[\mathcal{PT}, H] = 0$, i.e., $\mathcal{PT}H(\mathcal{PT})^{-1} = (\mathcal{PT})^{-1}H\mathcal{PT} = H$, where $\mathcal{P}$ and $\mathcal{T}$ are, respectively, the operators of parity (or space) and time-reversal transformations. These are defined according to $\mathcal{P}\,x\mathcal{P} = -x$, $\mathcal{P}\,p\mathcal{P} = \mathcal{T}p\mathcal{T} = -p$ and $\mathcal{T}iI\mathcal{T} = -iI$ where x, p, and I, are, respectively, the position, momentum and identity operatos acting on the Hilbert space $\mathcal{H} = L^2(\mathcal{R})$ and $i = \sqrt{-1}$. Note that this applies only for the system whose classical position x and momentum p are real [1, 2]. It is also known that $\mathcal{PT}-$symmetry

does not necessarily lead to completely real spectrum, and there are several potentials where part or all of the energy spectrums are complex. The Schrödinger equation (SE) for the real (Hermitian) potentials are investigated to generate the real energy eigenvalues which are of much interest [1]. Bender *et al.* for the first time and latter others have investigated several complex potentials on the $\mathcal{PT}$−symmetric quantum mechanics. The main reason for the growing recent interest in $\mathcal{PT}$−symmetry.[1] is that the eigenvalues (spectrum) of every Hamiltonian is real ($\mathcal{PT}$−symmetry is exact) or come in complex conjugate pairs of complex eigenvalues ($\mathcal{PT}$−symmetry is spontaneously broken) [1,2,3,4,5,6,7,8,9]. Afterwards, non-Hermitian Hamiltonians with real or complex spectra have been studied by using numerical and analytical techniques [10,11,12,13,14].

Various different techniques have been employed in solving the above mentioned potential cases. One of these methods which makes it possible to present the theory of special functions by starting from a differential equation has been developed by Nikiforov and Uvarov (NU) method [15]. This method is based on solving the time-independent SE by reducing it into a generalized equation of hypergeometric form. Exact solution of SE for central potentials has generated much interest in recent years. So far, these potentials are the parabolic type potential [16], the Eckart potential [17,18], the Fermi-step potential [17,18], the Rosen-Morse potential [19], the Ginocchio barrier [20], the Scarf barriers [21], the Morse potential [22] and a potential which interpolates between Morse and Eckart barriers [23]. Many authors have studied on exponential type potentials [12,24,25,26,27] and quasi exactly solvable quadratic potentials [28,29,30]. In addition, Dirac, Klein-Gordon, and Duffin-Kemmer-Petiau equations for a Coulomb type potential are solved by using this method [31,32]. The exact solutions for these models have been obtained analytically.

So far, we have solved the nonrelativistic and semi-relativistic wave equations using the statistical model [33], a different approach to the shifted large 1/N expansion technique [34] and also the shifted large $1/\bar{l}$ expansion technique [35] with a wide group of static phenomenological and QCD-motivated potentials to produce the heavy and light quarkonium spectra. In addition, the energy eigenvalues of the bound states of an electron in the gen-

eral exponential cosine screened Coulomb potential are obtained using the shifted large 1/N expansion method [36]. In this work, we solve the SE using NU method with some well-known WS potentials [4,13,14,37]. This potential form has been used widely in analysis of heavy-ion reaction and has enjoyed success [37]. It is selected for a shell model which can be used for describing metallic clusters in a successful way and for lighting the central part of the interaction neutron with one heavy nucleus [38,39].

In the present work, the energy eigenvalues and eigenfunctions of the Hermitian and non-Hermitian form of the generalized WS potential are calculated by employing the NU method.

The contents of this paper is as follows: In Section II we breifly present the generalized form of Woods-Saxon potential inspired from the SUSYQM. In Section III, we present Nikiforov-Uvarov method and also the solution of the Schrödinger equation with Hermitian form of the generalized WS potential for the $l = 0$ and $l \neq 0$ cases. In Section IV, the $\mathcal{PT}-$symmetric and non-$\mathcal{PT}-$symmetric non-Hermitian potential forms are also investigated. Shapes of the potential and energy are studied in Section V. Finally, in Section VI we give results and conclusions.

II. GENERALIZED WOODS-SAXON POTENTIAL

The motion of the free electrons which have conclusive influence on the abundance of metallic clusters is a vital problem in the nuclear physics. These electrons are moving in well defined orbitals, around the central nucleus and in a mean field potential which is produced by the positively charged ions and the rest of electrons. In the mean field potential, the details of the potential are described by free parameters such as depth, width and slope of the potential, which have to be fitted to experimental observations. The interactions between nuclei are commonly described by using a potential that consist of the Coulomb and the nuclear potentials. These potentials are usually taken to be of the ordinary WS form [40] and has been widely used in analysis of heavy-ion reactions [37]. Recently, Arai [41] has

introduced the deformed hyperbolic potentials using the definitions of the deformed hyper functions. As an example, we may choose the inter-nuclear generalized WS of spherically symmetric form as [42]

$$V(r) = -V_1 \frac{e^{-\left(\frac{r-R_0}{a}\right)}}{1+qe^{-\left(\frac{r-R_0}{a}\right)}} + V_2 \frac{e^{-2\left(\frac{r-R_0}{a}\right)}}{\left(1+qe^{-\left(\frac{r-R_0}{a}\right)}\right)^2}, \quad q \geq 1 \tag{1}$$

where r stands for the center-of-mass distance between the projectile nucleus and the target nucleus. The other parameters in the potential $R_0 = r_0 A^{1/3}$ is the radius of the corresponding spherical nucleus or the width of the potential, A is the target mass number, r_0 is the radius parameter, V_1 is the potential depth of the Coulombic part (i.e., the first term on the right hand side of Eq.(1)), a is the surface diffuseness parameter which is usually adjusted to the experimental values of ionization energies [37] and finally V_2 is an introduced parameter for the second part of Eq.(1) (transforms like potential barrier) [42]. Further, q is the deformation parameter for the strength of the exponential part other than unity which is arbitrarily taken to be a real constant within the potential. Its worthwhile to note that the spatial coordinates in the potential are not deformed and thus the potential still remains spherical. Therefore, we start with the original SE:

$$\left(\frac{p^2}{2m} + V(r)\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{2}$$

where the classical phase space is assumed to be real, i.e., x and p are the standard Hermitian operators representing the position and momentum of a particle of mass m. Employing the separation of variables

$$\psi(\mathbf{r}) = \frac{1}{r}R(r)Y(\theta, \phi), \tag{3}$$

leads to the simple radial SE, for all angular momentum states, of the type

$$-\frac{\hbar^2}{2m}\frac{d^2R(r)}{dr^2} + \left(V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right)R(r) = ER(r). \tag{4}$$

Now, the aim is to solve the last equation for the given potential in Eq.(1) for the $l = 0$ and $l \neq 0$ cases using the Nikiforov-Uvarov method which will be introduced breifly and employed in section III.

III. APPLYING NIKIFOROV-UVAROV METHOD TO THE SCHRÖDINGER EQUATION

The Nikiforov-Uvarov (NU) method provides us an exact solution of Eq.(4) for certain kind of potentials among them the one given in Eq.(1) [15]. This method is based upon the solutions of general second order linear differential equation with special orthogonal functions [43]. For a given real or complex potentials, the $\mathcal{PT}-$symmetric SE in one dimension is reduced to a generalized equation of hypergeometric type with an appropriate $s = s(x)$ coordinate transformation. Thus, it takes the form:

$$\psi''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{5}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\widetilde{\tau}(s)$ is of a first-degree polynomial. To find a particular solution for Eq. (5) by separation of variables, we use the transformation given by

$$\psi(s) = \phi(s)y(s). \tag{6}$$

This reduces SE, Eq.(5), into an equation of hypergeometric type:

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0, \tag{7}$$

where $\phi(s)$ is found to ısatisfy the condition $\phi'(s)/\phi(s) = \pi(s)/\sigma(s)$. Further, $y(s)$ is the hypergeometric type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{8}$$

where B_n is a normalizing constant and the weight function $\rho(s)$ must satisfy the condition [15]

$$(\sigma(s)\rho(s))' = \tau(s)\rho(s). \tag{9}$$

The function $\pi(s)$ and the parameter λ required for this method are defined as

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$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)}, \tag{10}$$

and

$$\lambda = k + \pi'(s). \tag{11}$$

Here, $\pi(s)$ is a polynomial with the parameter s and the determination of k is the essential point in the calculation of $\pi(s)$. Thus, for the determination of k, the discriminant under the square root is being set equal to zero and the resulting second-order polynomial has to be solved for its roots $k_{+,-}$. Hence, a new eigenvalue equation for the SE becomes

$$\lambda_n + n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s) = 0, \qquad (n = 0, 1, 2, ...) \tag{12}$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \tag{13}$$

and it will have a negative derivative. Therefore, we start solving SE for the $\mathcal{PT}$ generalized WS following the NU for $l = 0$ and $l \neq 0$ cases as follows:

A. Solution for the $l = 0$ case

In order to calculate the energy eigenvalues and the corresponding eigenfunctions, the Hermitian real-valued potential form given by Eq.(1) is substituted into the one−dimensional $\mathcal{PT}$−symmetrical SE with the zero angular momentum states,

$$R''(x) + \frac{2m}{\hbar^2}\left[E + \frac{V_1 e^{-\alpha x}}{1 + qe^{-\alpha x}} - \frac{V_2 e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2}\right]R(x) = 0, \tag{14}$$

where some assignments of one-dimensional parameter $x = r - R_0$ and $\alpha = 1/a$ are done.

Now, rewriting Eq.(14) by employing the convenient transformation, $s(x) = -e^{-\alpha x}$, then the $\mathcal{PT}$−symmetrical Hermitian one-dimensional SE becomes

$$\frac{d^2 R(s)}{ds^2} + \frac{1}{s}\frac{dR(s)}{ds} + \frac{2m}{\hbar^2\alpha^2 s^2}\left[E - \frac{V_1 s}{1 - qs} - \frac{V_2 s^2}{(1 - qs)^2}\right]R(s) = 0, \tag{15}$$

and also introducing the given dimensionless parameters:

$$\epsilon = -\frac{2mE}{\hbar^2 \alpha^2} > 0 \quad (E < 0), \quad \beta = \frac{2mV_1}{\hbar^2 \alpha^2} \quad (\beta > 0), \quad \gamma = \frac{2mV_2}{\hbar^2 \alpha^2} \quad (\gamma > 0), \qquad (16)$$

finally it leads into the following simple hypergeometric form given by

$$\frac{d^2 R(s)}{ds^2} + \frac{1 - qs}{s(1 - qs)} \frac{dR(s)}{ds} + \frac{1}{[s(1 - qs)]^2} \times \left[\left(-\epsilon q^2 + \beta q - \gamma \right) s^2 + (2\epsilon q - \beta) s - \epsilon \right] R(s) = 0.$$

$$(17)$$

Hence, comparing the last equation with the generalized hypergeometric type, Eq.(5), we obtain the associated polynomials as

$$\tilde{\tau}(s) = 1 - qs, \quad \sigma(s) = s(1 - qs), \quad \tilde{\sigma}(s) = \left(-\epsilon q^2 + \beta q - \gamma \right) s^2 + (2\epsilon q - \beta) s - \epsilon. \quad (18)$$

When these polynomials are substituted into Eq.(10), with $\sigma'(s) = 1 - 2qs$, we obtain

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2} \sqrt{(q^2 + 4\epsilon q^2 - 4\beta q + 4\gamma - 4kq) s^2 + 4 (\beta - 2\epsilon q + k) s + 4\epsilon}. \qquad (19)$$

Further, the discriminant of the upper expression under the square root has to be set equal to zero. Therefore, it becomes

$$\Delta = [4 (\beta - 2\epsilon q + k)]^2 - 4 \times 4\epsilon \left(q^2 + 4\epsilon q^2 - 4\beta q + 4\gamma - 4kq \right) = 0. \qquad (20)$$

Solving Eq.(20) for the constant k, we get the double roots as $k_{+,-} = -\beta \pm q \sqrt{\epsilon \left(1 + \frac{4\gamma}{q^2} \right)}$. Thus, substituting these values for each k into Eq.(19), we obtain

$$\pi(s) = -\frac{qs}{2} \pm \frac{1}{2} \begin{cases} \left[\left(2\sqrt{\epsilon} - \sqrt{1 + \frac{4\gamma}{q^2}} \right) qs - 2\sqrt{\epsilon} \right]; & \text{for} \quad k_+ = -\beta + q \sqrt{\epsilon \left(1 + \frac{4\gamma}{q^2} \right)}, \\ \left[\left(2\sqrt{\epsilon} + \sqrt{1 + \frac{4\gamma}{q^2}} \right) qs - 2\sqrt{\epsilon} \right]; & \text{for} \quad k_- = -\beta - q \sqrt{\epsilon \left(1 + \frac{4\gamma}{q^2} \right)}. \end{cases} \qquad (21)$$

Hence, making the following choice for the polynomial $\pi(s)$ as

$$\pi(s) = -\frac{qs}{2} - \frac{1}{2} \left[\left(2\sqrt{\epsilon} + \sqrt{1 + \frac{4\gamma}{q^2}} \right) qs - 2\sqrt{\epsilon} \right], \qquad (22)$$

gives the function:

$$\tau(s) = 1 - 2qs - \left[\left(2\sqrt{\epsilon} + \sqrt{1 + \frac{4\gamma}{q^2}} \right) qs - 2\sqrt{\epsilon} \right], \qquad (23)$$

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which has a negative derivative of the form $\tau(s) = -\left(2 + 2\sqrt{\epsilon} + \sqrt{1 + \frac{4\gamma}{q^2}}\right)q$. Thus, from Eq.(11) and Eq.(12), we find

$$\lambda = -\beta - \frac{q}{2}\left(1 + 2\sqrt{\epsilon}\right)\left(1 + \sqrt{1 + \frac{4\gamma}{q^2}}\right),\tag{24}$$

and

$$\lambda_n = \left(2 + 2\sqrt{\epsilon} + \sqrt{1 + \frac{4\gamma}{q^2}}\right)nq + n(n-1)q.\tag{25}$$

After setting $\lambda_n = \lambda$ and solving for ϵ, we find:

$$\epsilon_n(\alpha, q) = \left[\frac{1 + 2n}{2} - \frac{\left(n(n+1) - \frac{\beta}{q}\right)}{1 + 2n + \sqrt{1 + \frac{4\gamma}{q^2}}}\right]^2.\tag{26}$$

Therefore, substituting the values of ϵ, β and γ into Eq.(26) together with the transformation $\alpha = 1/a$, one can immediately determine the exact energy eigenvalues $E_n(a, q)$ as

$$E_n(a, q) = -\frac{\hbar^2}{2ma^2}\left[\frac{1 + 2n}{2} - \frac{\left(n(n+1) - \frac{2ma^2 V_1}{\hbar^2 q}\right)}{1 + 2n + \sqrt{1 + \frac{8ma^2 V_2}{\hbar^2 q^2}}}\right]^2, \quad 0 \leq n < \infty, \quad q \geq 1.\tag{27}$$

Following Ref.[36], in atomic units ($\hbar = m = c = e = 1$), Eq.(27) turns out to be

$$E_n(a, q) = -\frac{1}{2a^2}\left[\frac{1 + 2n}{2} - \frac{\left(n(n+1) - \frac{2a^2 V_1}{q}\right)}{1 + 2n + \sqrt{1 + \frac{8a^2 V_2}{q^2}}}\right]^2, \quad 0 \leq n < \infty, \quad q \geq 1.\tag{28}$$

The above equation indicates that we deal with a family of the generalized WS spherical potential. Of course it is clear that by imposing appropriate changes in the parameters a and V_1, the index n describes the quantization for the bound energy states. In addition, if the parameter V_2 in Eq.(28) is adjusted to zero (for s−state), solution reduces to the form obtained for the standard WS potential without regarding of the centrifugal barrier potential and the deformation parameter q in Eq.(4).[1]

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[1]The exact bound energy eigenvalues for the s−states, in atomic units, is $E_n(a = 1, q \to 1) = -\frac{1}{8}\left[n + 1 + \frac{2V_1}{(n+1)}\right]^2$, which is in agreement with previous works on SUSYQM (cf. formula (40) in Ref.[42]).

Let us now find the corresponding wavefunctions. Applying the NU method, the polynomial solutions of the hypergeometric function $y(s)$ depends on the determination of weight function $\rho(s)$ which is found to be

$$\rho(s) = (1 - qs)^{\eta-1} s^{2\sqrt{\epsilon}}; \quad \eta = 1 + \sqrt{1 + \frac{4\gamma}{q^2}}. \tag{29}$$

Substituting into the Rodrigues relation given in Eq.(8), the eigenfunctions are obtained in the following form

$$y_{n,q}(s) = C_n (1 - qs)^{-(\eta-1)} s^{-2\sqrt{\epsilon}} \frac{d^n}{ds^n} \left[(1 - qs)^{n+\eta-1} s^{n+2\sqrt{\epsilon}} \right], \tag{30}$$

where C_n stands for the normalization constant and its value is $1/n!$. In the limit $q \to 1$, the polynomial solutions of $y_n(s)$ are expressed in terms of Jacobi Polynomials, which is one of the classical orthogonal polynomials, with weight function $(1 - qs)^{\eta-1} s^{2\sqrt{\epsilon}}$ in the closed interval $[0, 1]$, yielding $y_{n,1}(s) \simeq P_n^{(2\sqrt{\epsilon}, \eta-1)}(1 - 2s)$ [43]. Finally, the other part of the wave function in Eq.(6) (with $q \to 1$) is found to be

$$\phi(s) = (1 - s)^\mu s^{\sqrt{\epsilon}}, \quad \mu = \eta/2. \tag{31}$$

Combining the Jacobi polynomials and $\phi(s)$ in Eq.(31), the s−wave functions ($l = 0$) could be determined as

$$R_n(s) = D_n s^{\sqrt{\epsilon}} (1 - s)^{\eta/2} P_n^{(2\sqrt{\epsilon}, \eta-1)}(1 - 2s), \tag{32}$$

with $s = -e^{-\alpha x}$ and D_n is a new normalization constant.

B. Solution for the $l \neq 0$ case

The Hamiltonian for the generalized WS potential for the $l \neq 0$ case is

$$H = \frac{p^2}{2m} - V_1 \frac{e^{-\left(\frac{r-R_0}{a}\right)}}{1 + qe^{-\left(\frac{r-R_0}{a}\right)}} + \frac{l(l+1)\hbar^2}{2mr^2}. \tag{33}$$

To evaluate the spectra of energy eigenvalues and eigenfunctions we introduce a new effective potential whose form is given as:

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$$V_{eff} = -V_1 \frac{e^{-\left(\frac{r-R_0}{a}\right)}}{1 + qe^{-\left(\frac{r-R_0}{a}\right)}} + \frac{l(l+1)\hbar^2\alpha^2}{2m} \frac{e^{-2\left(\frac{r-R_0}{a}\right)}}{\left(1 + qe^{-\left(\frac{r-R_0}{a}\right)}\right)^2}, \tag{34}$$

where the second term of the last expression looks like the potential barrier term of Eq.(33). Thus, comparing Eq.(34) with its counterpart Eq.(1), one can arrive into the transformation of $V_2 \to \frac{l(l+1)\hbar^2\alpha^2}{2m}$. Moreover, Eq.(34) can also be rewriten as

$$V_{eff} = -V_1 \frac{1}{e^{\left(\frac{r-R_0}{a}\right)} + q} + \frac{l(l+1)\hbar^2\alpha^2}{2m\left(e^{\left(\frac{r-R_0}{a}\right)} + q\right)^2}, \tag{35}$$

where $\alpha = (q+1)/R_0$ and $\alpha = 1/a$. The lowest energy levels of the potential in Eq.(34) are now given

$$E_{n,l}(a,q) = -\frac{\hbar^2}{2ma^2} \left[\frac{1+2n}{2} - \frac{\left(n(n+1) - \frac{2ma^2V_1}{\hbar^2 q}\right)}{1 + 2n + \sqrt{1 + \frac{4l(l+1)}{q^2}}}\right]^2, \quad 0 \le n < \infty, \quad q \ge 1. \tag{36}$$

For scattering processes, it has been well accepted that the surface diffuseness parameter a is around 0.63 fm [37,44,45]. Much larger diffuseness parameter, ranging between 0.8 and 1.4 fm is needed in order to fit the data [37,46].

IV. NON-HERMITIAN POTENTIAL FORMS

A. $\mathcal{PT}$-symmetric and non-Hermitian generalized WS case

For $\mathcal{PT}-$symmetric and non-Hermitian potential case, we take the potential parameters, in Eq.(1), V_1, V_2, and q as real and $\alpha \to i\alpha_I$ is purely imaginary. In this case, the potential turns out to be

$$V(x) = -V_1 \left(\frac{q + \cos\alpha_I x - i\sin\alpha_I x}{1 + q^2 + 2q\cos\alpha_I x}\right) + V_2 \left(\frac{q + \cos\alpha_I x - i\sin\alpha_I x}{1 + q^2 + 2q\cos\alpha_I x}\right)^2. \tag{37}$$

By substituting this potential into Eq.(14) and making similar operations in obtaining Eq.(28), one can easily get the energy eigenvalues, in atomic units, as

$$E_n(\alpha_I, q) = \frac{1}{2} \left[\frac{\alpha_I(1+2n)}{2} - \frac{\left(n(n+1)\alpha_I + \frac{2V_1}{\alpha_I q}\right)}{1 + 2n + \sqrt{1 - \frac{8V_2}{\alpha_I^2 q^2}}}\right]^2, \tag{38}$$

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and

$$\lambda = \beta - \frac{q}{2}\left(1 + i2\sqrt{\epsilon}\right)\left(1 + \sqrt{1 - \frac{8mV_2}{\hbar^2\alpha_I^2 q^2}}\right), \tag{39}$$

$$\lambda_n = 2\left(1 + i\sqrt{\epsilon} + \frac{1}{2}\sqrt{1 - \frac{8mV_2}{\hbar^2\alpha_I^2 q^2}}\right)nq + n(n-1)q. \tag{40}$$

Here α_I is an arbitrary real parameter and $i = \sqrt{-1}$. Thus, by choosing the parameter α as purely imaginary, we find the energy eigenvalues obtained for $\mathcal{PT}$−symmetric and non-hermitian generalized WS potential are not similar to Eq.(28). A positive energy spectra is obtained if and only if $n < \sqrt{\frac{2m}{\hbar^2\alpha_I^2}\left(\frac{V_1}{q} - \frac{V_2}{q^2}\right)} - \frac{1}{2}\left(1 + \sqrt{1 - \frac{8mV_2}{\hbar^2\alpha_I^2 q^2}}\right)$, since the energy eigen-values of generalized WS potential are negative.[2] It appears that the eigenvalues are always positive real for $V_2 = 0$, that is, $E_n(\alpha_I, q) = \frac{1}{2}\left[\frac{1+n}{2}\alpha_I - \frac{V_1}{(1+n)\alpha_I q}\right]^2$, but can be complex for $V_2 > \frac{\alpha_I^2 q^2}{8}$. Since these one–dimensional non-Hermitian Hamiltonian were invariant under $\mathcal{PT}$−transformation, they possessed real spectra. Thus, their real spectral properties were linked with their $\mathcal{PT}$−symmetry.

On the other hand, following the procedures in Section III, we obtain an eigenfunction for the non-Hermitian potential, Eq.(37), as:

$$R_n(s) = D_n s^{\widetilde{E}_n}(1-s)^{\frac{1}{2}\left(1+\sqrt{1-8V_2/\alpha_I^2}\right)}P_n^{(2\widetilde{E}_n,\sqrt{1-8V_2/\alpha_I^2})}(1-2s), \tag{41}$$

with $\widetilde{E}_n = \left[\frac{1+2n}{2} - \frac{\left(n(n+1)+\frac{2V_1}{\alpha_I^2}\right)}{1+2n+\sqrt{1-\frac{8V_2}{\alpha_I^2}}}\right]$ and $s = -e^{-i\alpha_I x}$. Its clear that the eigenfunctions are always complex for all values of V_2. Therefore, if $V_2 = 0$ (with $q \to 1$), we have a simple wavefunction taking the form:

$$R_n(x) = D_n(-1)^{\widetilde{E}_n}e^{-i\alpha_I \widetilde{E}_n x}(1 + e^{-i\alpha_I x})P_n^{(2\widetilde{E}_n,1)}(1 + 2e^{-i\alpha_I x}), \tag{42}$$

with $\widetilde{E}_n = \left(\frac{1+n}{2} - \frac{V_1}{(1+n)\alpha_I^2}\right)$.

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[2]Once we set $V_2 = 0$, then $n < \sqrt{\frac{2mV_1}{\hbar^2\alpha_I^2 q}} - 1$ which agrees with Ref.[13,14].

B. Non-$\mathcal{PT}$-symmetric and non-Hermitian generalized WS case

Another form of the potential is obtained by making the selection of the potential parameter as pure imaginary, that is, V_1 is replaced by iV_{1I}, and α is replaced by $i\alpha_I$ but V_2 remains a pure real. In this case, the Hamiltonian is non-Hermitian and non-$\mathcal{PT}$−symmetric having real spectra. The potential turns out to be

$$V(x) = -V_{1I}\left(\frac{\sin\alpha_I x + i(q + \cos\alpha_I x)}{1 + q^2 + 2q\cos\alpha_I x}\right) + V_2\left(\frac{q + \cos\alpha_I x - i\sin\alpha_I x}{1 + q^2 + 2q\cos\alpha_I x}\right)^2. \tag{43}$$

By substituting this potential into Eq.(14) and making similar operations in obtaining Eq.(28), one can easily get the energy eigenvalues, in atomic units, as

$$E_n(\alpha_I, q) = \frac{1}{2}\left[\frac{\alpha_I(1 + 2n)}{2} - \frac{\left(i\frac{2V_{1I}}{\alpha_I q} + n(n + 1)\alpha_I\right)}{1 + 2n + \sqrt{1 - \frac{8V_2}{\alpha_I^2 q^2}}}\right]^2, \tag{44}$$

and

$$\lambda = i\beta - \frac{q}{2}\left(1 + i2\sqrt{\epsilon}\right)\left(1 + \sqrt{1 - \frac{8mV_2}{\hbar^2\alpha_I^2 q^2}}\right), \tag{45}$$

$$\lambda_n = 2\left(1 + i\sqrt{\epsilon} + \frac{1}{2}\sqrt{1 - \frac{8mV_2}{\hbar^2\alpha_I^2 q^2}}\right)nq + n(n - 1)q. \tag{46}$$

The last case has real plus imaginary energy spectra. When we consider the real part of energy eigenvalues an acceptable result is obtained when $n < \sqrt{\frac{2m}{\hbar^2\alpha_I^2}\left(\frac{iV_{1I}}{q} - \frac{V_2}{q^2}\right)} - \frac{1}{2}\left(1 + \sqrt{1 - \frac{8mV_2}{\hbar^2\alpha_I^2 q^2}}\right)$ condition. However, the energy spectrum is not seen at the imaginary part of energy eigenvalues, since it is independent of n.[3]

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[3]Once we set $V_2 = 0$, then $n < \sqrt{\frac{i2mV_1}{\hbar^2\alpha_I^2 q}} - 1$ which agrees with Ref.[13,14].

V. POTENTIAL AND ENERGY SHAPES

It is illustrated in Figure 1, the shape of the generalized WS potential given by Eq.(1) for various $q = 1, 3$ and 7 values. This is done by fixing the potential parameters $V_1 = 50 \ MeV$, $V_2 = 10 \ MeV$, $r_0 = 1.285 \ fm$, $A = 56$ and $a = 0.65 \ fm$ [37,44,45]. The empirical values found by Perey et al. [47] $r_0 = 1.285 \ fm$, $a = 0.65 \ fm$ and $A = 56$ which is the geometric average of the target nucleus mass number $44 \leq A \leq 72$ [48] are used to draw the potential in Figure 2 for various values of $V_2 = 10, 50, 100 \ MeV$. Further, Figure 3 shows the energy eigenvalues in atomic units as a function of the discrete level n for various $a = 0.65, 0.85, 1.05 \ fm$ values. Of course, it is clear that by imposing appropriate changes in the parameter a, the index n describes the quantization of the bound states and energy spectrum. Some of the initial energy levels for undeformed case ($q = 1$) value, are presented by choosing $V_1 = 50 \ MeV$, $V_2 = 10 \ MeV$ in Figure 4. Its worthwhile to note that Figure 3 and Figure 4 have no lower bound on the spectrum for both $V_2 = 0$ and $V_2 \neq 0$ cases, respectively. In this regard, it is found that for $V_2 = 0$ case, the energy levels go into lower bound for $0 < n < 6$ states and into higher bound for $n > 6$ states. However, for $V_2 \neq 0$ case, the energy levels go into lower bound for $0 < n < 2$ states and higher bound for $n > 2$ states.

VI. RESULTS AND CONCLUSIONS

In this article we used NU method and solved the radial SE for the generalized WS potential with the angular momentum $l = 0$ ($V_2 = 0$) and $l \neq 0$ ($V_2 \neq 0$). A particularly interesting result of our investigation is that all the $\mathcal{PT}$−symmetric Hamiltonians with potential parameters remain all purely real have a real bound energies E_n with $n \geq 0$ for Hermitian case and real positive in contrary to expectation if one lets $\alpha \rightarrow i\alpha_I$ in the generalized WS potential. Therefore, for non-Hermitian case, the spectrum is real for $V_2 = 0$ but complex conjugate for some values of $V_2 \neq 0$ in the generalized WS potential. Further, when α and V_1 parameters are purely complex, it is seen that the number of

discrete levels for bound states is given only by the real part of energy eigenvalues. Thus, for a $\mathcal{PT}-$symmetric Hamiltonians the exactness of $\mathcal{PT}-$symmetry implies the reality of spectrum. More specifically, if an eigenfunction $R_{n,l}(s)$ is a $\mathcal{PT}-$invariant, $\mathcal{PT}\ R_{n,l}(s) = R_{n,l}(s)$, then the corresponding eigenvalue of $E_{n,l}$ is real. The exact $\mathcal{PT}-$symmetry is a sufficient condition. But for a given $\mathcal{PT}-$symmetric Hamiltonian, it is not easy to determine the exactness of $\mathcal{PT}-$symmetry without actually solving the corresponding radial SE. In this regard, the wave functions are physical and energy eigenvalues are in good agreement with the results obtained by the other methods [42].

On the other hand, the effect of the centrifugal barrier potential which goes as $1/r^2$ was replaced by a term having exactly the original WS form but of a second degree in order to reproduce its effect. This $l \neq 0$ term has its physical basis arising from the superpotential partner of the WS potential in SUSYQM [42]. Hence, this new barrier term retakes the exact form of the original potential but with a small perturbed strength factor; say V_2 [42,49]. According to the complex quantum mechanics [50], the eigenvalues of the conversion $\alpha \rightarrow i\alpha_I$ are not simultanously eigenstates of $\mathcal{PT}-$operator.

Finally, we point out that the exact results obtained for the generalized WS potential may have some interesting applications in the study of different quantum mechanical systems and nuclear scattering.

FIGURES

FIG. 1. Variation of the generalized WS potential as a function r. The curves are shown for various values of the deformation parameter q

FIG. 2. Variation of the generalized WS potential as a function r. The curves are shown for various values of the perturbed parameter V_2.

FIG. 3. The variation of the energy eigenvalues with respect to the quantum number n with $V_1 = 50 \ MeV$ and $V_2 = 0$. The curves shown are for various values of the surface diffuseness parameter a.

FIG. 4. The variation of the energy eigenvalues with respect to the quantum number n with $V_1 = 50 \ MeV$ and $V_2 = 10 \ MeV$. The curves shown are for various values of the surface diffuseness parameter a.

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